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University Of Diyala – College Of Engineering

Digital Signal Processing

Course Instructor
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Lecture No. 6: Z-Transform

Third Class
Department of Computer and Software Engineering

http://www.engineering.uodiyala.edu.iq/
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Lecture Outline

- What are Transforms?
- Why do we Use Transforms?
- What are Benefits of Transforms?
- Z-Transforms.
- Region of Convergence.
- Properties of Z-Transforms.
- Inverse Z-Transform.

What are Transforms

- The term ***transform*** refers to a mathematical operation that takes a given function, called the ***original*** and returns a new function, referred to as the ***image***
- The transformation is often done by an integral or summation formula
- Commonly used transforms are named after Laplace and Fourier

What Do We Use Transforms

- Transforms are used to change a complicated problem into a simpler one:
  1. The simpler problem is solved in the image domain
  2. By using the inverse transform we obtain the solution in the original domain
- Examples:
  1. Laplace transform to solve a **differential equation**
  2. z transform to solve a **difference equation**

Benefits of Transforms

- Transforms are used to examine nature of signals or sequences
- They are helpful to solve LTI systems by transforming differential or difference equations into algebraic equations

$$
\begin{aligned}
y_1 &= x(t) \\
y_2 &= y_1 + y_5 + y_6 \\
y_3 &= \omega \int_0^t y_2(\tau)\,\mathrm{d}\tau \\
y_4 &= \omega \int_0^t y_3(\tau)\,\mathrm{d}\tau \\
y_5 &= -\frac{1}{Q} y_3 \\
y_6 &= -y_4
\end{aligned}
$$

$$\Longrightarrow$$

$$
\begin{aligned}
Y_1(s) &= X(s) \\
Y_2(s) &= Y_1(s) + Y_5(s) + Y_6(s) \\
Y_3(s) &= \omega \frac{Y_2(s)}{s} \\
Y_4(s) &= \omega \frac{Y_3(s)}{s} \\
Y_5(s) &= -\frac{1}{Q} Y_3(s) \\
Y_6(s) &= -Y_4(s)
\end{aligned}
$$

Z-Transforms

- Laplace transforms: are used as a tool for solving continuous-time linear time-invariant systems and electric circuits.
- Z-transforms: are used as a tool for solving discrete-time linear time-invariant systems

We'll only focus on z-transforms in DSP

Z-Transforms

- For discrete time signals we usually compute z - Transform to analyze the signal in frequency domain.
  - z - Transform makes it easier to analyze the signal and also reduces the complexity in calculations.
- It plays an important role in analyzing **causal** systems specified by linear constant-coefficient **difference** equations

Definition of Z-Transforms

Definition:

The z-transform of a discrete time signal x[n] is defined as:

$$\boxed{X(Z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}}$$

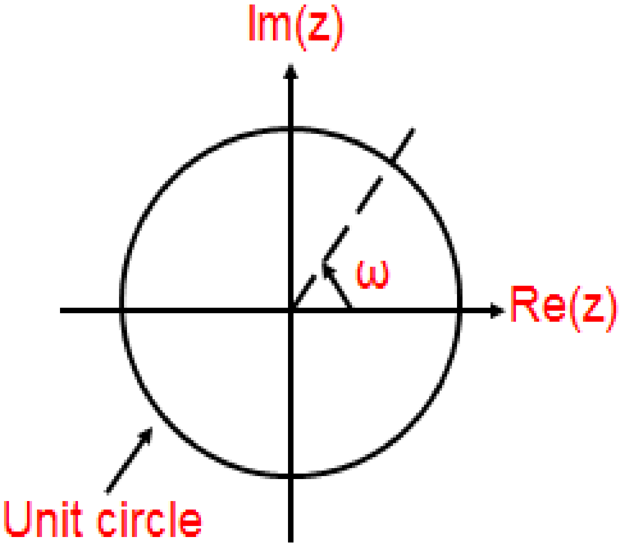
Where z = $r.e^{j\omega}$ is a complex variable.

Notationaly, if x[n] has z-transform, we write

$$x[n] \xleftrightarrow{Z} H(z)$$

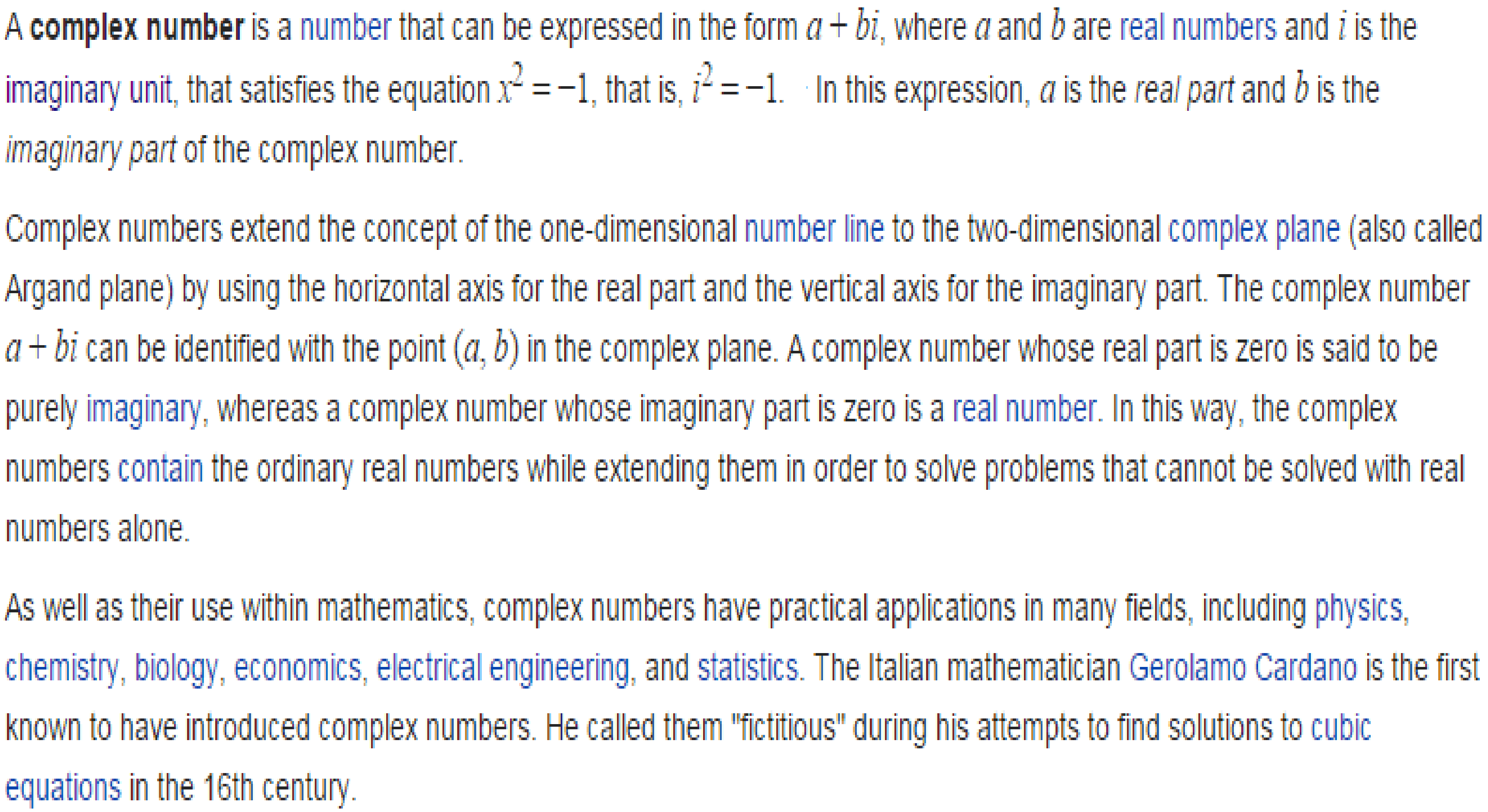
z-transform may be viewed as the DTFT of an Exponentially weighted sequence.

$$X(Z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} [r^n x(n)] e^{-jn\omega}$$



Unit circle in the complex z-plane

Complex Numbers

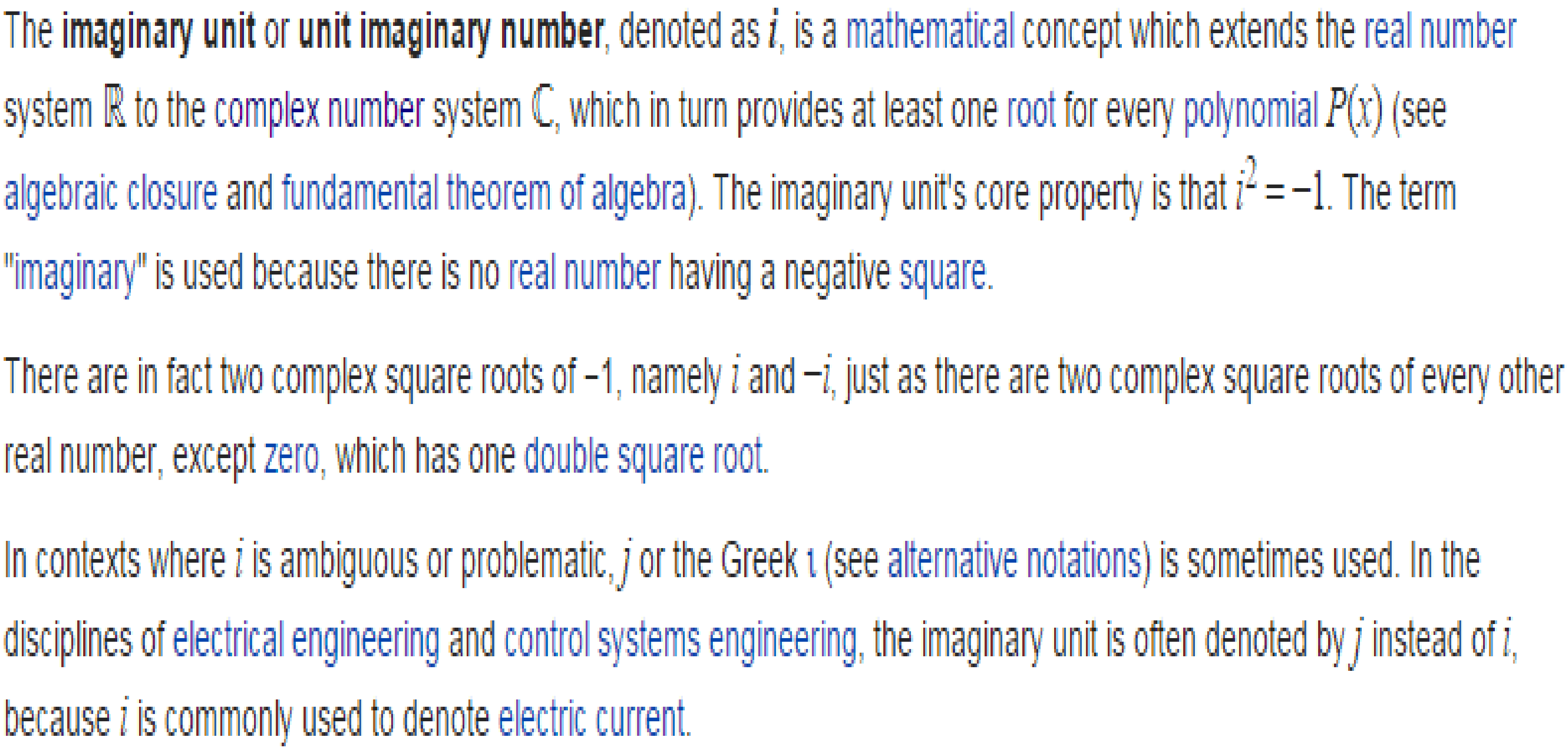


A **complex number** is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $x^2 = -1$, that is, $i^2 = -1$. In this expression, a is the *real part* and b is the *imaginary part* of the complex number.

Complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane (also called Argand plane) by using the horizontal axis for the real part and the vertical axis for the imaginary part. The complex number $a + bi$ can be identified with the point (a, b) in the complex plane. A complex number whose real part is zero is said to be purely imaginary, whereas a complex number whose imaginary part is zero is a real number. In this way, the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

As well as their use within mathematics, complex numbers have practical applications in many fields, including physics, chemistry, biology, economics, electrical engineering, and statistics. The Italian mathematician Gerolamo Cardano is the first known to have introduced complex numbers. He called them "fictitious" during his attempts to find solutions to cubic equations in the 16th century.

Complex Numbers

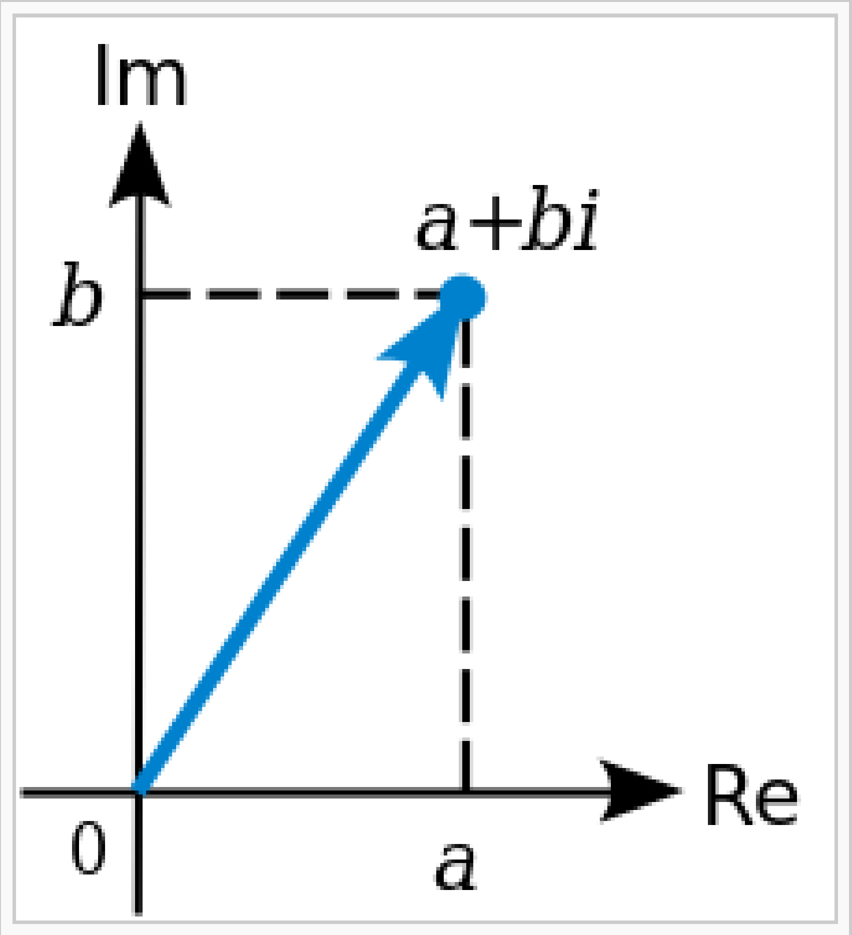


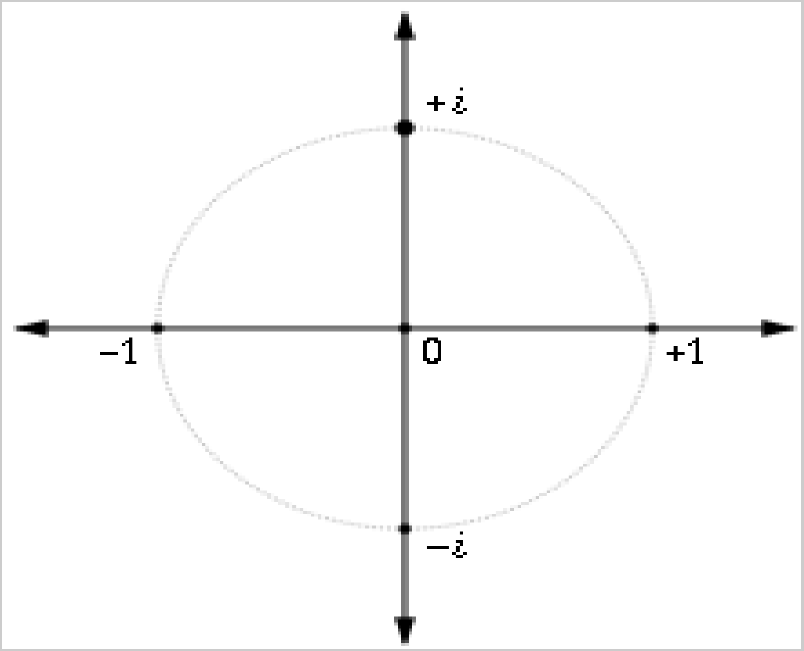
The **imaginary unit** or **unit imaginary number**, denoted as i, is a mathematical concept which extends the real number system $\mathbb{R}$ to the complex number system $\mathbb{C}$, which in turn provides at least one root for every polynomial $P(x)$ (see algebraic closure and fundamental theorem of algebra). The imaginary unit's core property is that $i^2 = -1$. The term "imaginary" is used because there is no real number having a negative square.

There are in fact two complex square roots of −1, namely i and $-i$, just as there are two complex square roots of every other real number, except zero, which has one double square root.

In contexts where i is ambiguous or problematic, j or the Greek ι (see alternative notations) is sometimes used. In the disciplines of electrical engineering and control systems engineering, the imaginary unit is often denoted by j instead of i, because i is commonly used to denote electric current.

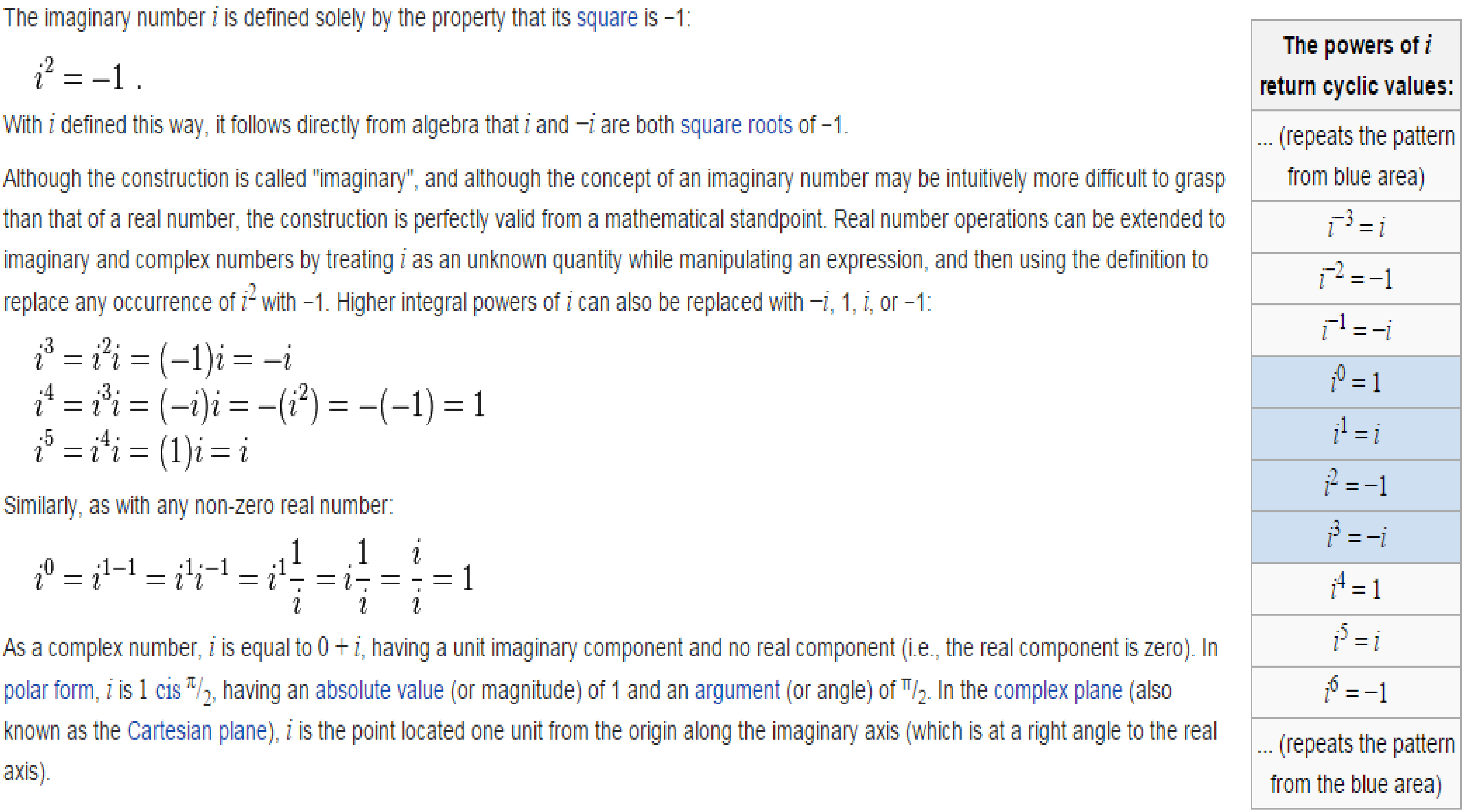
Complex Numbers





i in the complex or cartesian plane. Real numbers lie on the horizontal axis, and imaginary numbers lie on the vertical axis

Complex Numbers



The imaginary number i is defined solely by the property that its square is −1:

$$i^2 = -1\ .$$

With i defined this way, it follows directly from algebra that i and $-i$ are both square roots of −1.

Although the construction is called "imaginary", and although the concept of an imaginary number may be intuitively more difficult to grasp than that of a real number, the construction is perfectly valid from a mathematical standpoint. Real number operations can be extended to imaginary and complex numbers by treating i as an unknown quantity while manipulating an expression, and then using the definition to replace any occurrence of i^2 with −1. Higher integral powers of i can also be replaced with $-i$, 1, i, or −1:

$$\begin{aligned} i^3 &= i^2 i = (-1)i = -i \\ i^4 &= i^3 i = (-i)i = -(i^2) = -(-1) = 1 \\ i^5 &= i^4 i = (1)i = i \end{aligned}$$

Similarly, as with any non-zero real number:

$$i^0 = i^{1-1} = i^1 i^{-1} = i^1 \frac{1}{i} = i\frac{1}{i} = \frac{i}{i} = 1$$

As a complex number, i is equal to $0 + i$, having a unit imaginary component and no real component (i.e., the real component is zero). In polar form, i is 1 cis $^{\pi}/_{2}$, having an absolute value (or magnitude) of 1 and an argument (or angle) of $^{\pi}/_{2}$. In the complex plane (also known as the Cartesian plane), i is the point located one unit from the origin along the imaginary axis (which is at a right angle to the real axis).

The powers of i return cyclic values:
... (repeats the pattern from blue area)
$i^{-3} = i$
$i^{-2} = -1$
$i^{-1} = -i$
$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$
... (repeats the pattern from the blue area)

Complex Numbers

$$(a+bi)+(c+di)=(a+c)+(b+d)i.$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

$$(a+bi)(c+di)=(ac-bd)+(bc+ad)i.$$

$$i^2=i\times i=-1.$$

$(a+bi)(c+di)=ac+bci+adi+bidi$ (distributive law)

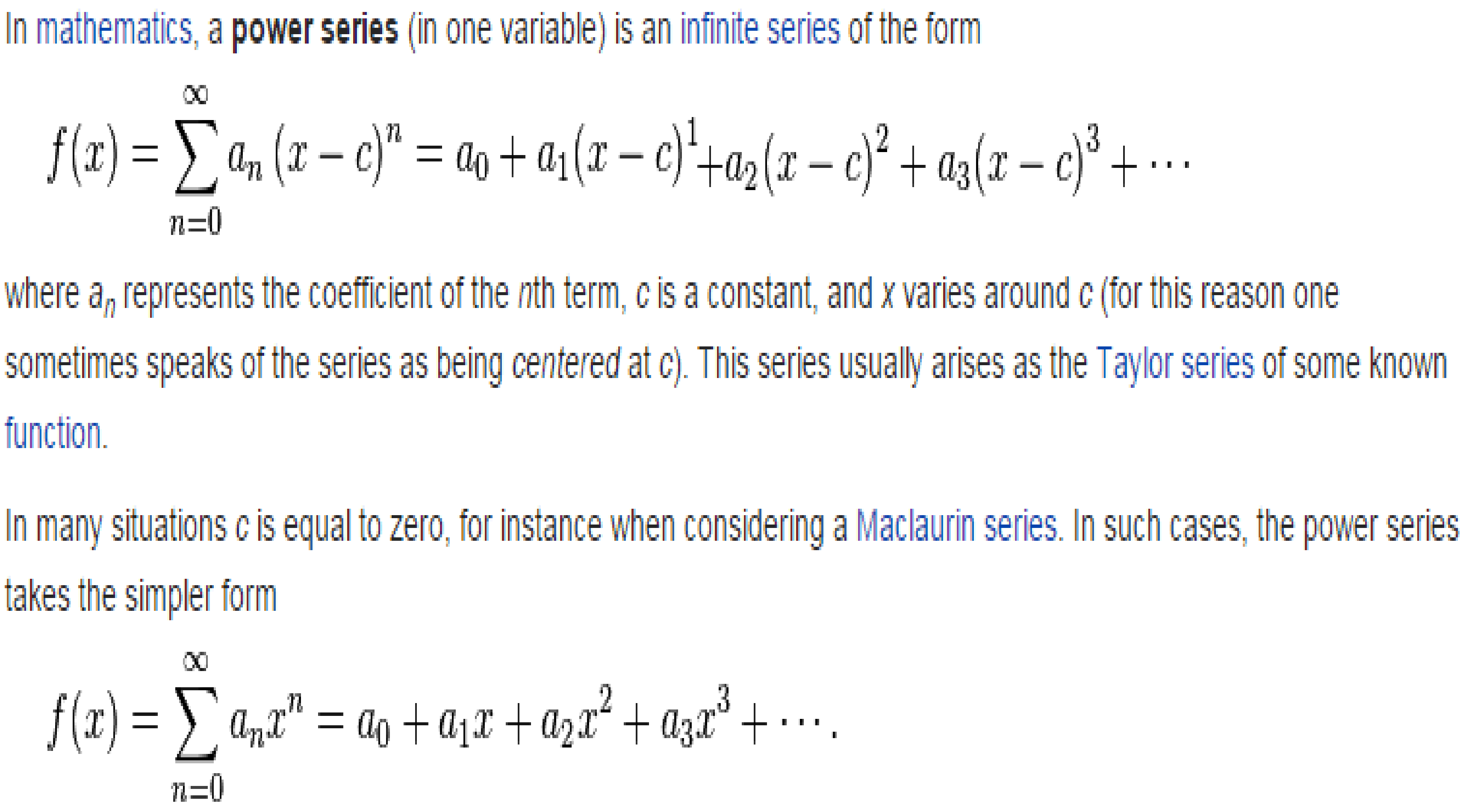
$=ac+bidi+bci+adi$ (commutative law of addition—the order of the summands can be changed)

$=ac+bdi^2+(bc+ad)i$ (commutative and distributive laws)

$=(ac-bd)+(bc+ad)i$ (fundamental property of the imaginary unit).

$$\frac{a+bi}{c+di}=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i.$$

Power Series



In mathematics, a **power series** (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x - c\right)^n = a_0 + a_1(x - c)^1 + a_2(x - c)^2 + a_3(x - c)^3 + \cdots$$

where a_n represents the coefficient of the *n*th term, *c* is a constant, and *x* varies around *c* (for this reason one sometimes speaks of the series as being *centered* at *c*). This series usually arises as the Taylor series of some known function.

In many situations *c* is equal to zero, for instance when considering a Maclaurin series. In such cases, the power series takes the simpler form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Power Series

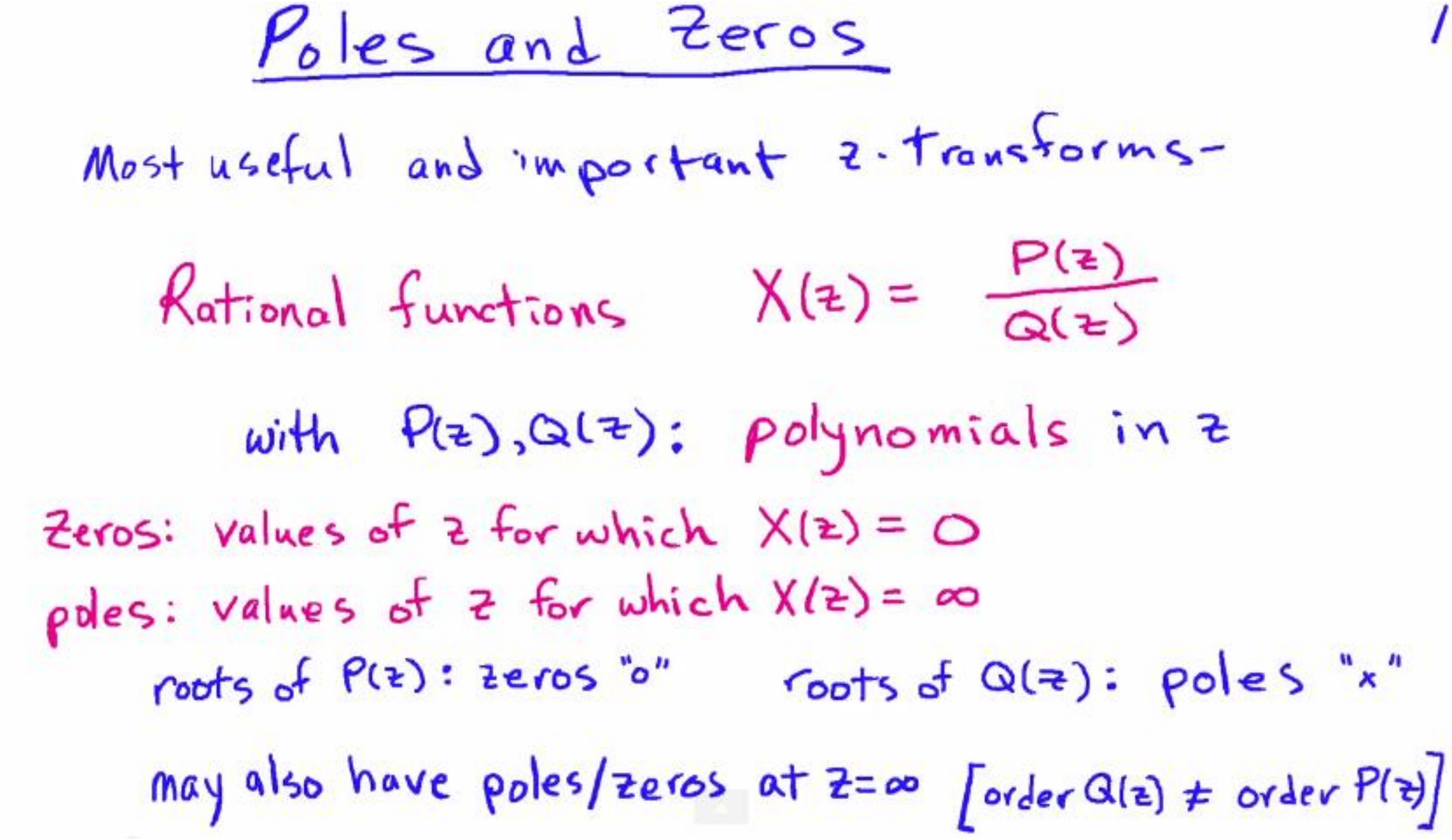
$$f(x) = 3 + 2x + 1x^2 + 0x^3 + 0x^4 + \cdots$$

$$f(x) = 6 + 4(x-1) + 1(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

Poles and Zeros of Z-Transform



Poles and Zeros

Most useful and important z-transforms -

Rational functions $X(z) = \frac{P(z)}{Q(z)}$

with $P(z), Q(z)$: polynomials in z

Zeros: values of z for which $X(z) = 0$

poles: values of z for which $X(z) = \infty$

roots of $P(z)$: zeros "o"   roots of $Q(z)$: poles "x"

may also have poles/zeros at $z=\infty$ [order $Q(z) \neq$ order $P(z)$]

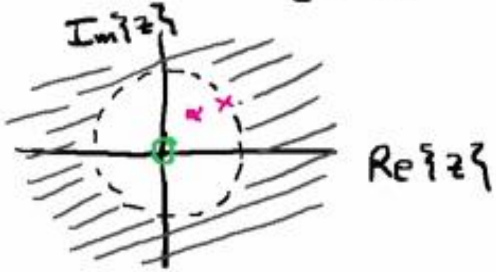
Poles and Zeros of Z-Transform

Examples:

1) $x[n] = \alpha^n u[n] \xleftrightarrow{z} X(z) = \dfrac{z}{z-\alpha}, \quad |z| > |\alpha|$

zero @ $z = 0$

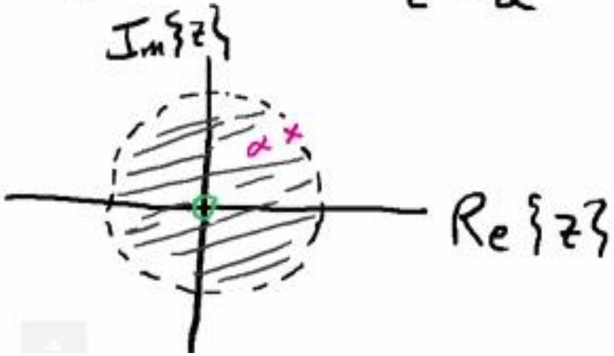
pole @ $z = \alpha$



2) $x[n] = -\alpha^n u[-n-1] \xleftrightarrow{z} X(z) = \dfrac{z}{z-\alpha}, \quad |z| < |\alpha|$

zero @ $z = 0$

pole @ $z = \alpha$



Poles and Zeros of Z-Transform

Examples-

3) $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow}$

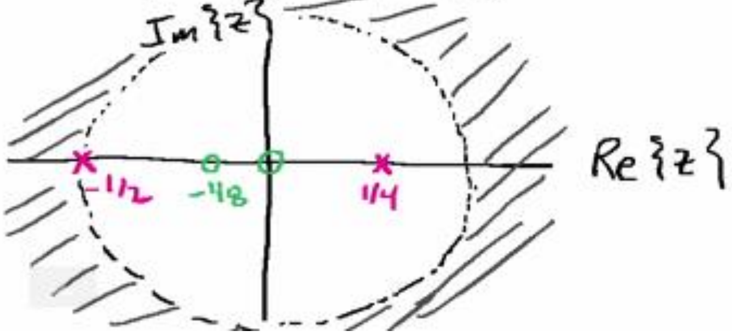
$$X(z) = \frac{z}{z-1/4} + \frac{z}{z+1/2}, \quad |z| > 1/2$$

$$X(z) = \frac{z^2 + 1/2 z + z^2 - 1/4 z}{(z-1/4)(z+1/2)}, \quad |z| > 1/2$$

$$= \frac{2z^2 + 1/4 z}{(z-1/4)(z+1/2)} = \frac{2z(z+1/8)}{(z-1/4)(z+1/2)}, \quad |z| > 1/2$$

zeros @ $z = 0, -1/8$

poles @ $z = 1/4, -1/2$



Poles and Zeros of Z-Transform

Examples-

4) $x[n] = (1/4)^n u[n] - (-\frac{1}{2})^n u[-n-1] \xleftrightarrow{z}$

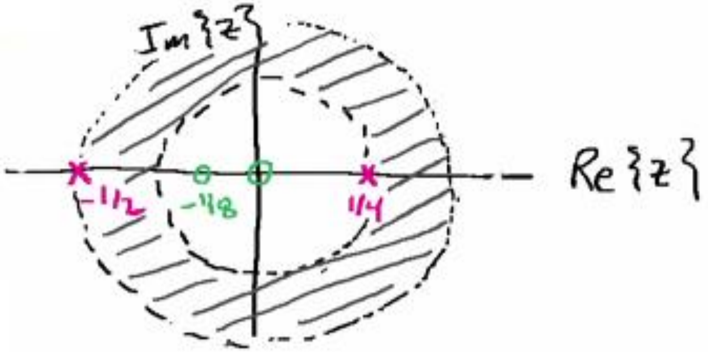
$$X(z) = \underbrace{\frac{z}{z-1/4}}_{|z|>1/4} + \underbrace{\frac{z}{z+1/2}}_{|z|<1/2}, \quad 1/4 < |z| < 1/2$$

from Ex. 3)

$$X(z) = \frac{2z(z+1/8)}{(z-1/4)(z+1/2)}, \quad 1/4 < |z| < 1/2$$

zeros @ $z = 0, -1/8$

poles @ $z = 1/4, -1/2$



Poles and Zeros of Z-Transform

Examples: 5) $x[n] = \begin{cases} \alpha^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} (\alpha z^{-1})^n = \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}} = \frac{z^N - \alpha^N}{z^{N-1}(z-\alpha)}$$

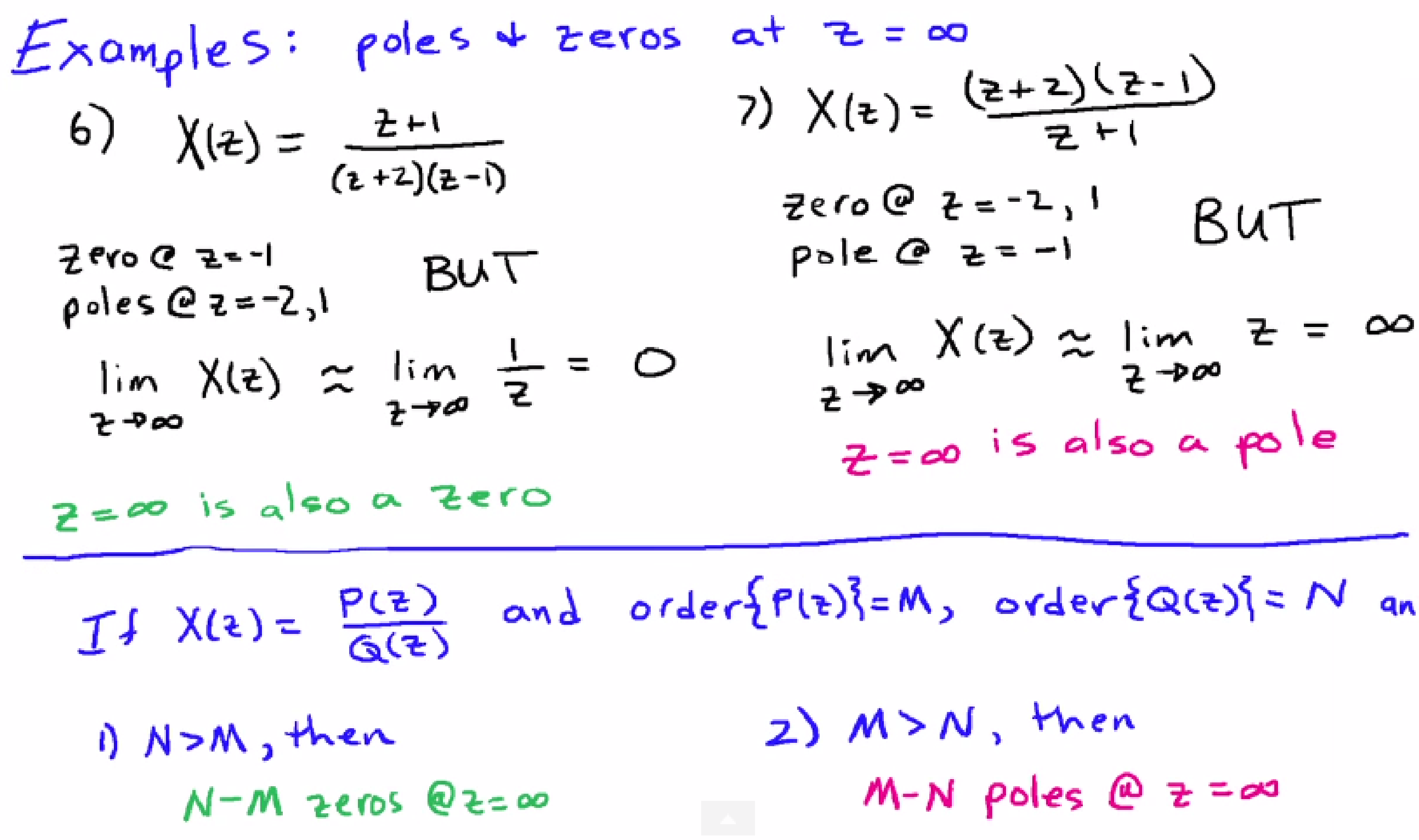
ROC: $\sum_{n=0}^{N-1} |\alpha z^{-1}|^n < \infty \Rightarrow |\alpha| < \infty$ and $z \neq 0$

(entire z-plane, except $z=0$)

Zeros: $z^N = \alpha^N \Rightarrow z_k = \alpha e^{j\frac{2\pi}{N}k}$, $k = \cancel{0}, 1, 2 \cdots N-1$

poles: $z=0$ $(N-1)$ and $\boxed{z=\alpha}$ — cancels out with zero $z_0 = \alpha$

Poles and Zeros of Z-Transform



Examples: poles & zeros at $z = \infty$

6) $X(z) = \frac{z+1}{(z+2)(z-1)}$

zero @ $z=-1$

poles @ $z=-2,1$

BUT

$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} \frac{1}{z} = 0$

$z=\infty$ is also a zero

7) $X(z) = \frac{(z+2)(z-1)}{z+1}$

zero @ $z=-2,1$

pole @ $z=-1$

BUT

$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} z = \infty$

$z=\infty$ is also a pole

If $X(z) = \frac{P(z)}{Q(z)}$ and $\text{order}\{P(z)\}=M$, $\text{order}\{Q(z)\}=N$ and

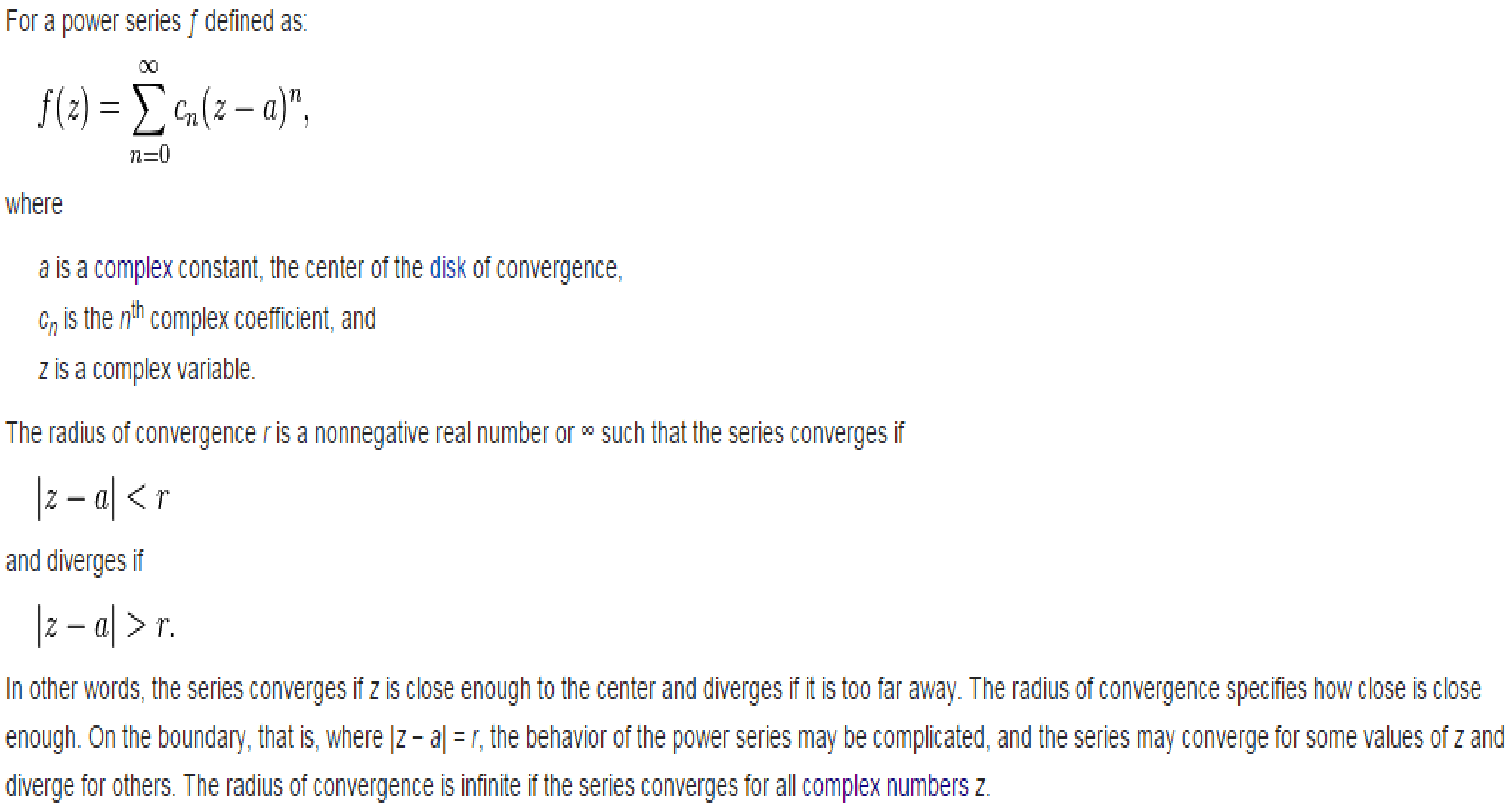
1) $N>M$, then

$N-M$ zeros @ $z=\infty$

2) $M>N$, then

$M-N$ poles @ $z=\infty$

Radius of Convergence



For a power series *f* defined as:

$$f(z)=\sum_{n=0}^{\infty} c_n (z-a)^n,$$

where

a is a complex constant, the center of the disk of convergence,

c_n is the n^{th} complex coefficient, and

z is a complex variable.

The radius of convergence *r* is a nonnegative real number or ∞ such that the series converges if

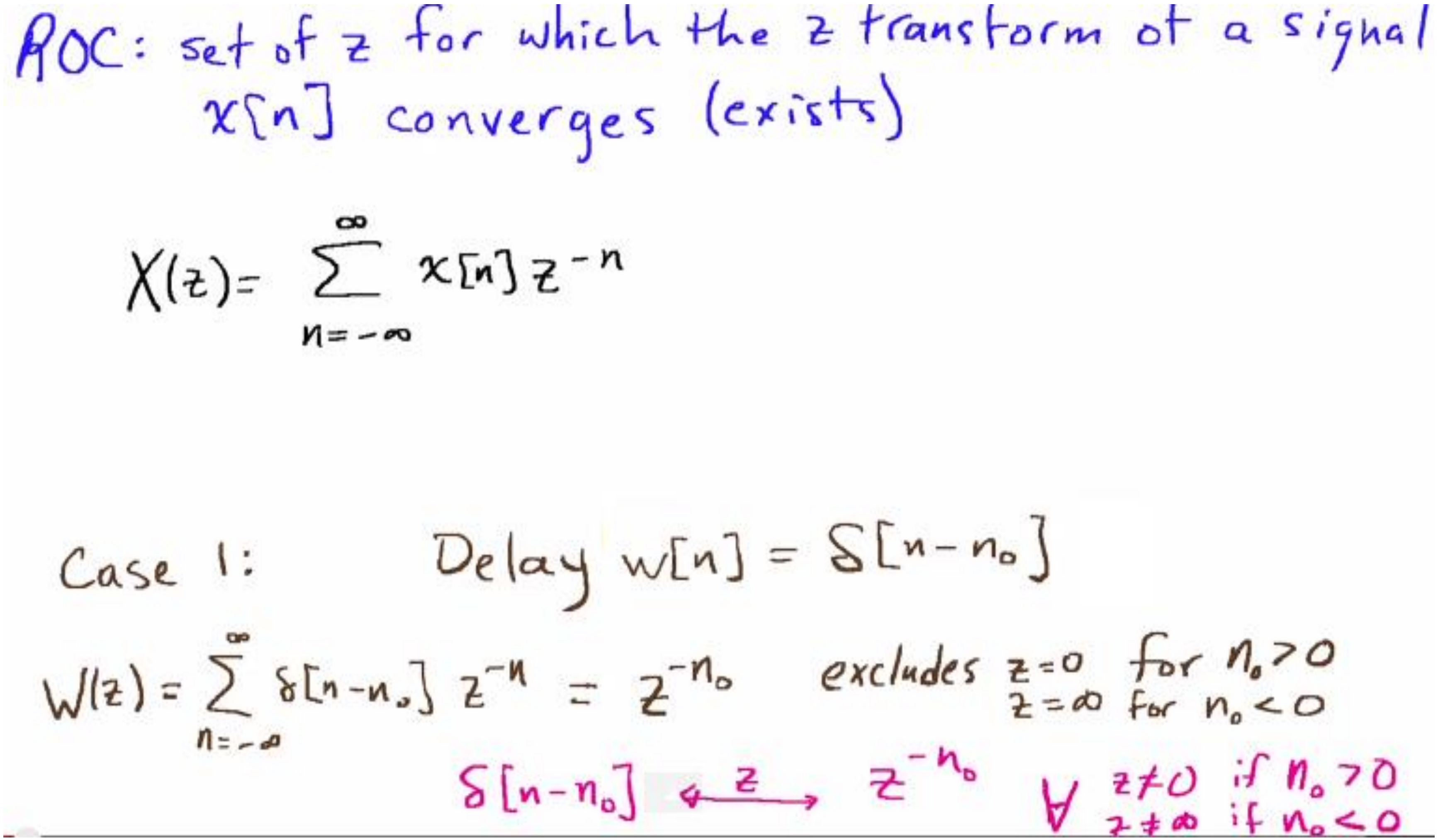
$$|z-a|<r$$

and diverges if

$$|z-a|>r.$$

In other words, the series converges if z is close enough to the center and diverges if it is too far away. The radius of convergence specifies how close is close enough. On the boundary, that is, where $|z - a| = r$, the behavior of the power series may be complicated, and the series may converge for some values of z and diverge for others. The radius of convergence is infinite if the series converges for all complex numbers z.

Region of Convergence (ROC)



ROC: set of z for which the z transform of a signal x[n] converges (exists)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Case 1: Delay $w[n] = \delta[n-n_0]$

$$W(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_0]\, z^{-n} = z^{-n_0}$$

excludes $z=0$ for $n_0>0$

$z=\infty$ for $n_0<0$

$$\delta[n-n_0] \xleftrightarrow{z} z^{-n_0} \quad \forall \begin{cases} z\neq 0 & \text{if } n_0>0 \\ z\neq\infty & \text{if } n_0<0 \end{cases}$$

Region of Convergence (ROC)

Case 2: $x[n] = \alpha^n u[n]$

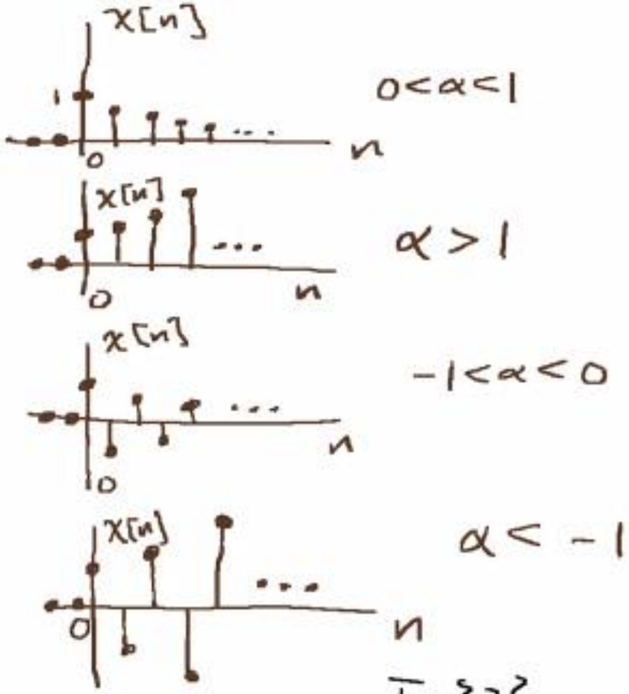
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} \qquad \text{provided } |\alpha z^{-1}| < 1$$

$$|z| > |\alpha|$$

if $|\alpha z^{-1}| \geq 1$, $X(z)$ does not converge

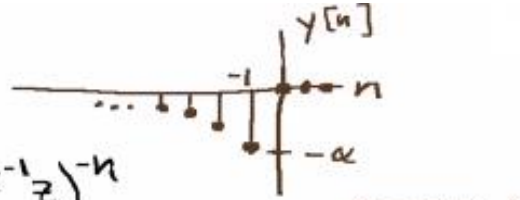


$$\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad |z|>|\alpha|$$

ROC: $|z|>|\alpha|$

Region of Convergence (ROC)

Case 3: $y[n] = -\alpha^n u[-n-1]$



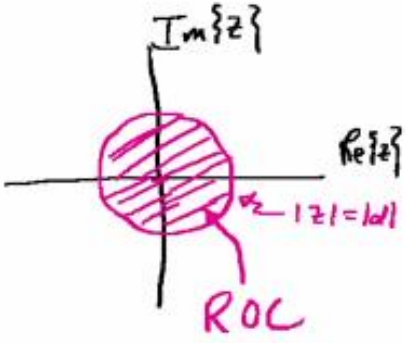
$0<\alpha<1$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{n=-\infty}^{-1} (\alpha^{-1} z)^{-n}$$

$$= -\sum_{l=1}^{\infty} (\alpha^{-1} z)^{l} = -\sum_{l=0}^{\infty} (\alpha^{-1} z)^{l} + 1 = \frac{-1}{1-\alpha^{-1} z} + 1 \quad \text{for } |\alpha^{-1} z| < 1$$

$$= \frac{\alpha}{z-\alpha} + 1 = \frac{z}{z-\alpha} \quad \left(\text{or } \frac{1}{1-\alpha z^{-1}}\right) \qquad |z| < |\alpha|$$

$$-\alpha^n u[-n-1] \xleftrightarrow{z} \boxed{\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}} \quad |z| < |\alpha|$$



X(z) (Case 2) is identical to Y(z)!

Differ in the ROC - z transforms nonunique w/o ROC

Region of Convergence (ROC)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

power series in z

Convergence ⇒ absolutely summable

Require $\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} < \infty$

ROC depends on $|z|$ - circles/rings in z-plane

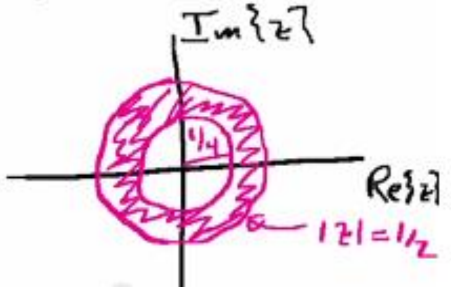
Example: $g[n] = (1/4)^n u[n] - (1/2)^n u[-n-1]$

$(1/4)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-1/4}, \quad |z| > 1/4$

$-(1/2)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z-1/2}, \quad |z| < 1/2$

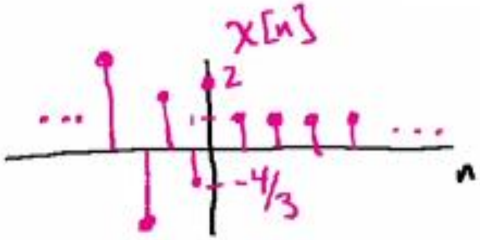
$G(z) = \frac{z}{z-1/4} + \frac{z}{z-1/2}, \quad 1/4 < |z| < 1/2$

$= \frac{2z^2 - 3/4 z}{(z-1/4)(z-1/2)},$



Region of Convergence (ROC)

Example: $x[n] = u[n] + (-3/4)^n u[-n]$



To use transform pairs, rewrite

$$x[n] = u[n] + \delta[n] - -(-3/4)^n u[-n-1]$$

$$u[n] \xleftrightarrow{z} \frac{z}{z-1}, \boxed{|z| > 1}$$

$$\delta[n] \xleftrightarrow{z} 1$$

$$-(-3/4)^n u[-n-1] \xleftrightarrow{z} \frac{z}{z+3/4}, \boxed{|z| < 3/4}$$

incompatible

X(z) does not exist!

Region of Convergence (ROC)

Region of Convergence:

Region of Convergence (ROC) is a set of all values of z, for which X(z) attains a finite value. It defines the region for which z-Transform exists.

Since, the z-Transform is a power series, it attains finite value or converges when it is absolutely summable. Stated differently:

$$|X(z)| = \sum_{n=0}^{\infty} (|x[n]z^{-n}|) < \infty$$

We will later on illustrate this concepts by some simple examples.

Properties of ROC:

- The ROC cannot contain any poles.
    - By definition a pole exists when X(z) is infinite. Since X(z) must be finite for all z for convergence, there cannot be a pole in the ROC.
- If x[n] is a finite-duration sequence, then the ROC is the entire z-plane, except possibly $|z| = 0$ or $|z| = \infty$.
    - A finite-duration sequence is a sequence that is nonzero in a finite interval $n_1<n<n_2$. As long as each value of x [n] is finite then the sequence will be absolutely summable.
        - When $n_2>0$ there will be az^{-1} term and thus the ROC will not include $z = 0$.
        - When $n_2<0$ then the sum will be infinite and thus the ROC will not include $|z| = \infty$.
        - On the other hand, when $n_2 \leq 0$ then the ROC will include $z = 0$, and when $n_1 \geq 0$ the ROC will include $|z| = \infty$.

Region of Convergence

As noted above, the z-Transform converges when |X(z)| < ∞, then we can split the infinite sum into positive and negative-time portions. So,

$$|X(Z)| \leq P(Z) + N(Z)$$

Where $N(Z) = \sum_{n=-\infty}^{-1} \left(| x[n] z^{-n} | \right)$ and $P(Z) = \sum_{n=0}^{\infty} \left(| x[n] z^{-n} | \right)$

In order for X(Z) to be finite, |x[n]| must be bounded. Let us now set

$$|x[n]| \leq C_1 r_1^{-n} \quad \text{for n<0}$$

and $$|x[n]| \leq C_2 r_2^{-n} \quad \text{for n≥0}$$

Region of Convergence

From this some further properties can be derived:

- If x[n] is a right-sided sequence, then the ROC extends outwards from the outermost pole in X(z).
  - A right-sided sequence is a sequence where $x[n] = 0$ for $n < n_1 < \infty$. Looking at the positive-time portion from the above derivation, it follows that

$$P(Z) = C_2 \sum_{n=0}^{\infty} \left(r_2^n z^{-n} \right) = C_2 \sum_{n=0}^{\infty} \left(\left(\frac{r_2}{z} \right)^n \right)$$

  - Thus in order for this sum to converge, $|z| > r_2$, and therefore the ROC of a right-sided sequence is of the form $|z| > r_2$.
- If x[n] is a left-sided sequence, then the ROC extends inwards from the innermost pole in X(z)
  - A left-sided sequence is a sequence where $x[n] = 0$ for $n > n_2 > -\infty$. Looking at the negative-time portion from the above derivation, it follows that

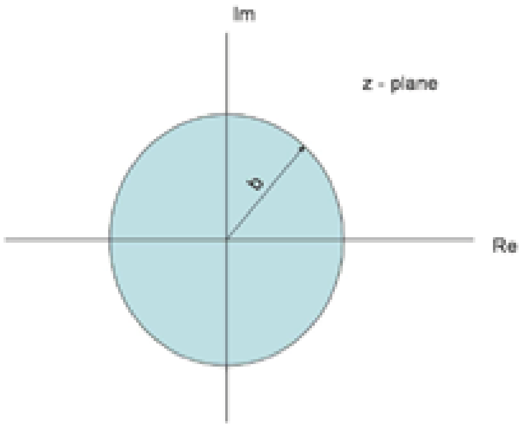
$$N(Z) = C_1 \sum_{n=-\infty}^{-1} \left(r_1^n z^{-n} \right) = C_1 \sum_{n=-\infty}^{-1} \left(\left(\frac{r_1}{z} \right)^n \right) = C_1 \sum_{k=1}^{\infty} \left(\left(\frac{z}{r_1} \right)^k \right)$$

  - Thus in order for this sum to converge, $|z| < r1$, and therefore the ROC of a left-sided sequence is of the form $|z| < r1$.

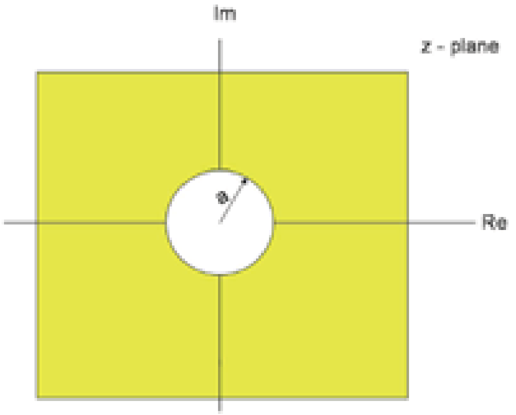
Region of Convergence

- If x [n] is a two-sided sequence, the ROC will be a ring in the z-plane that is bounded on the interior and exterior by a pole.
  - A two-sided sequence is a sequence with infinite duration in the positive and negative directions.
- From the derivation of the above two properties, it follows that if $r_2 < r < r_1$ which is common region where both sums are finite, converges thus X (z) converges as well. Therefore, the ROC of a two-sided sequence is of the form $r_1 < |z| < r_2$.
- If $r_2 > r_1$, there is no common region of convergence for the two sums and hence X(z) does not exist.

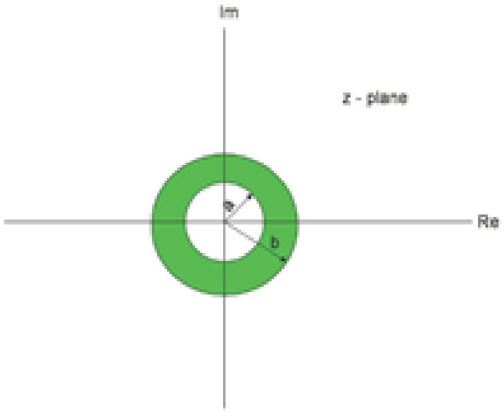
Region of Convergence



ROC for Left-Sided Sequence



ROC for Right-Sided Sequence



ROC for |X(z)|

Examples

- **Example 1**: Determine the z-transform of the following signals

(a)$x[n] = [1, \ 2, \ 5, \ 7, \ 0, \ 1]$

Solution: $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5},$ **ROC: entire z plane except z = 0**

(b) $y[n] = [1, 2, \underline{5}, 7, 0, 1]$

Solution: $Y(z) = z^{2} + 2z + 5 + 7z^{-1} + z^{-3},$ **ROC: entire z-plane except z=0 & ∞.**

(c)$z[n] = [0, 0, 1, 2, 5, 7, 0, 1]$

Solution: $z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7},$ **ROC: all z except z=0**

Examples

Example 2: Determine the z-transform of $x[n] = (1/2)^n u[n]$

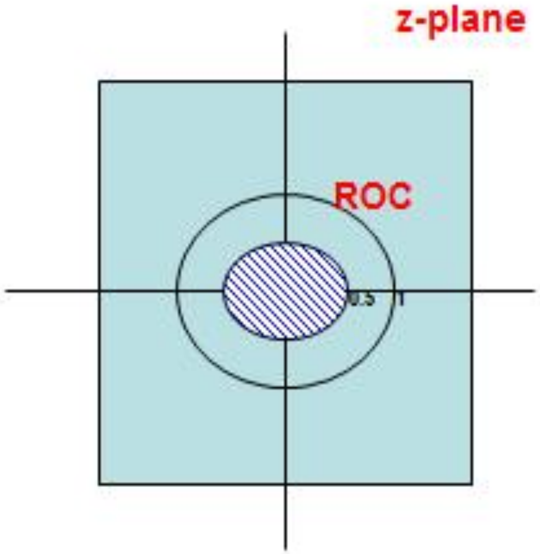
Solution 2:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2} z\right)^{-2} + \ldots\ldots$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}}$$



ROC: $|1/2\, z^{-1}| < 1$, or equivalently $|z| > 1/2$

Examples

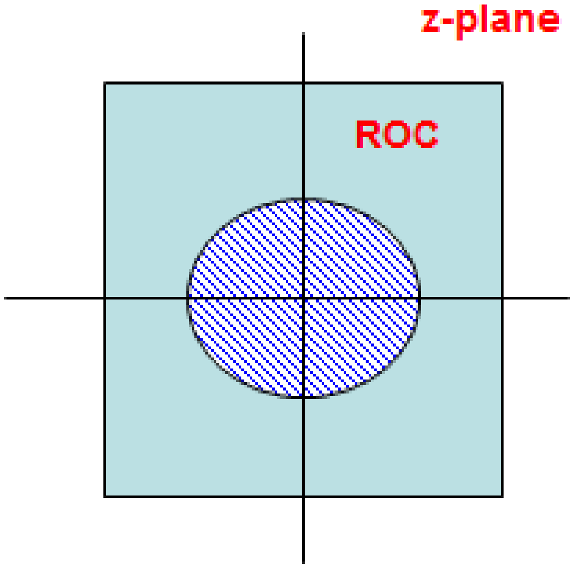
Example 3: Determine the z-transform of x[n] = $a^n u[n]$

Solution 3:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$= 1 + az^{-1} + \left(az^{-1}\right)^2 + \ldots\ldots$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$



Here, X(z) is taken to be converging, i.e, $|az^{-1}|<1$ or $|z|>a$. Hence, for the signal x[n] the RoC is entire region outside the circle $z = ae^{j\omega}$

Examples

Example 4: Find out the z-transform of unit impulse sequence.

Solution 4: The unit impulse has only one term that is equal to one when n = 0; therefore

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n).z^{-n} = 1.z^{-0} = 1.1 = 1$$

ROC: entire Z plane

Example 5: Find out the z-transform of unit step function.

Solution 5: From the definition of step function and z-transform we get,

$$Z\{u(n)\} = \sum_{n=0}^{\infty} 1.z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + ... = \frac{1}{1 - z^{-1}}$$

ROC: $|z^{-1}|<1$ or $|z|>1$

Tutorial

Tutorial 1:

1- Find the z-Transform and ROC of the following:

(a) Unit ramp signal $x[n]=n$

(b) $q[n] = \delta[n - k], \ k > 0$

(c) $r[n] = \delta[n+k], \ k > 0$

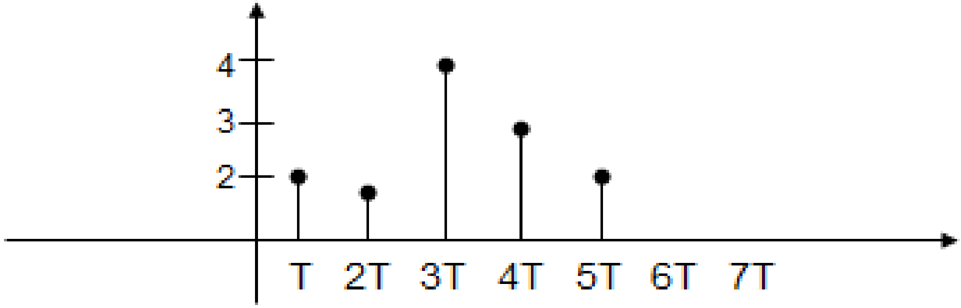
2- Find the z-transform of:

(a) $x(n) = e^{j\theta n}u(n)$

(b) $x(n) = rn \cos(\theta n)u(n)$, where $0 < r \leq 1$

(c) $x(n) = \mathrm{Sin}(\theta n)u(n)$

3- Find the z-transform of the finite length signal shown in the figure below.



Z-Transform Pairs

$$\delta_{(n)} \leftrightarrow 1$$

$$u_{(n)} \leftrightarrow \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$a^{n} u_{(n)} \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$n\, u_{(n)} \leftrightarrow \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$\sin(\theta n)\, u_{(n)} \leftrightarrow \frac{z \sin(\theta)}{z^2 - 2z\cos(\theta) + 1}$$

$$\cos(\theta n)\, u_{(n)} \leftrightarrow \frac{z(z - \cos(\theta))}{z^2 - 2z\cos(\theta) + 1}$$

$$e^{-\alpha n} \sin(\theta n)\, u_{(n)} \leftrightarrow \frac{z\, e^{-\alpha} \sin(\theta)}{z^2 - 2z\, e^{-\alpha} \cos(\theta) + e^{-2\alpha}}$$

$$e^{-\alpha n} \cos(\theta n)\, u_{(n)} \leftrightarrow \frac{z(z - e^{-\alpha} \cos(\theta))}{z^2 - 2z\, e^{-\alpha} \cos(\theta) + e^{-2\alpha}}$$

Properties of Z-Transform

P1- Linearity:

If $x_1[n] \leftrightarrow X_1(z)$

and $x_2[n] \leftrightarrow X_2(z)$

then

$$\boxed{a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(z) + a_2X_2(z)}$$

Properties of Z-Transform

Example 1: Determine the z-transform of $x[n]=[3(2^n)-(3^n)]u[n]$

Solution 1:

$$\because z[a^n u[n]] = \frac{1}{1-az^{-1}}$$

$$\therefore z[3(2)^n - 4(3)^n] = 3\frac{1}{1-2z^{-1}} - 4\frac{1}{1-3z^{-1}}$$

Example 2: Determine the z-transform of the signal $(\cos w_0 n)u[n]$

Soultion 2:

$$\because [\cos w_0 n]u[n] = \frac{1}{2}e^{jw_0 n} + \frac{1}{2}e^{-jw_0 n}$$

$$\therefore z\{[\cos w_0 n]u[n]\} = \frac{1}{2}\frac{1}{1-e^{jw_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jw_0}z^{-1}}$$

$$= \frac{1-z^{-1}\cos w_0}{1-2z^{-1}\cos w_0 + z^{-2}}$$

Properties of Z-Transform

P2- Time Shifting Property:

If $x[n] \leftrightarrow X(z)$

then $\boxed{x[n-k] \leftrightarrow z^{-k}X(z)}$

Proof: Since $z[x[n-k]] = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n}$

then the change of variable m = n-k produces

$$z[x[n-k]] = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+k)}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-k}X(z)$$

Properties of Z-Transform

Example 3: Use time shifting property to find z-transform of u[n]–u[n-N].

Soultion 3: The z-transform of u[n] can be found as

$$z[u[n]] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \ldots\ldots = \frac{1}{1 - z^{-1}}$$

Now the z-transform of u[n]-u[n-N] may be found as follows:

$$z[u[n] - u[n-N]] = \frac{1}{1 - z^{-1}} - z^{-N} \frac{1}{1 - z^{-1}}$$

$$= \frac{1 - z^{-N}}{1 - z^{-1}}$$

Properties of Z-Transform

P3- Scaling in the z-domain:

If $x[n] \leftrightarrow X(z)$

Then $\boxed{a^n x[n] \leftrightarrow X(a^{-1}z),}$ for any constant a, real or complex.

Proof: $$z\left[a^n x[n]\right] = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(a^{-1}z\right)^{-n} = X\left(a^{-1}z\right)$$

Example 4: Determine the z-transform of $a^n(\cos w_0 n)u[n]$.

Solution 4: Since

$$z[\cos(w_0 n)u[n]] = \frac{1 - z^{-1}\cos w_0}{1 - 2z^{-1}\cos w_0 + z^{-2}}$$

$$\therefore z[a^n(\cos w_0 n)u[n]] = \frac{1 - az^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}}$$

Properties of Z-Transform

P4- Time reversal:

If $x[n] \leftrightarrow X(z)$ then $\boxed{X[-n] \leftrightarrow X(z^{-1})}$

Proof:

$$z[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{m} = \sum_{m=-\infty}^{\infty} x[m] \left(z^{-1}\right)^{-m} = X(z^{-1})$$

Example 5: Determine the z-transform of u[-n].

Solution 5: Since $z[u[n]] = 1/(1 - z^{-1})$.

Therefore, $Z[u[-n]] = 1/(1-z)$

Properties of Z-Transform

P5- Differentiation in the z – Domain:

$x[n] \leftrightarrow X(z)$ then $\boxed{nx[n] = -z(dX(z)/dz)}$

Example 6: Determine the z-transform of the signal **$x[n] = na^n u[n]$**.

Solution 6:

$$\because z[a^n u[n]] = \frac{1}{1 - az^{-1}}$$

$$\therefore z[na^n u[n]] = -z \frac{d}{dz} \frac{1}{1 - az^{-1}} = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2}$$

Convolution in Z-Transform

If $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ then

$$\boxed{x_1[n]*x_2[n] = X_1(z)X_2(z)}$$

Proof: The convolution of $x_1[n]$ and $x_2[n]$ is defined as

$$x[n] = x_1[n]*x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

The z-transform of x[n] is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right]z^{-n}$$

Upon interchanging the order of the summation and applying the time shifting property, we obtain

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}\right] = X_2(z)\sum_{k=-\infty}^{\infty} x_1[k]z^{-k} = X_2(z)X_1(z)$$

Convolution in Z-Transform

Example 7: The following example illustrates the relation between the z-Transform and convolution. Consider two sequences: $x_1[n]=[1\ \ 2\ 2]$ and $x_2[n]=[2\ \ 3\ 4]$

The Z transform of $x_1(n)$ is.

$X_1(z)=1 + 2z^{-1} + 2z^{-2}$

The Z transform of $x_2(n)$ is.

$X_2(z)=2 + 2z^{-1} + 4z^{-2}$

Multiplying the 2 transforms give X(z), the Z Transform of the convolved signal.

$X(z) = X1(z)X2(z)=(\ 1 + 2z^{-1} + 2z^{-2})(\ 2 + 2z^{-1} + 4z^{-2}\ )$

$X(z) = 2 + 7z^{-1} +14z^{-2} +14z^{-3} + 8z^{-4}$

Taking the inverse z-Transform gives the following for the convolution of the 2 sampled signals.

$x(n) = [2\ 7\ 14\ 14\ 8]$

Convolution in Z-Transform

Instead of using the z-Transforms, we can convolve the two signals directly using the convolution summation as illustrated below.

$$x(n) = \sum_{k=0}^{\infty} x_1(k) x_2(n-k)$$

In this sum m must range over all values for which the product is finite. If both signals have a total of m samples (6 for this case) then there must be m - 1 values for n (5 in this case).

The following equations shows the convolution sum being evaluated for values of n from 0 to 4 (5 terms). Only the non-zero contributions are included.

$$x(0)=x_1(0)x_2(0)=2$$

$$x(1)=x_1(0)x_2(1)+x_1(1)x_2(0)=7$$

$$x(2)=x_1(0)x_2(2)+x_1(1)x_2(1)+x_1(2)x_2(0)=14$$

$$x(3)=x_1(1)x_2(2)+x_1(2)x_2(1)=14$$

$$x(4)=x_1(2)x_2(2)=8$$

Hence $x(n) = [2\ 7\ 14\ 14\ 8]$. The above equations show that applying the convolution sum directly give the same result as multiplying the two Z Transforms and taking the inverse transform.

Convolution in Z-Transform

Example 8: Compute the convolution of the signals $x_1[n]$ = [1, -2, 1] and

$$x_2[n]=\begin{cases}1, & 0\leq n\leq 5\\ 0, & elsewher\end{cases}$$

Solution 8:

$X_1(z) = 1 – 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

Now $X(z) = X_1(z)X_2(z) = 1 – z^{-1} – z^{-6} + z^{-7}$

Hence $x[n]$ = [1, -1, 0, 0, 0, 0, -1, 1]

Note: You should verify this result from the definition of the convolution sum.

Correlation in Z-Transform

If $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ then

$$\boxed{C_{x1x2}[k] = X_1(z)X_2(z^{-1})}$$

Initial Value Theorem:

If x[n] is causal then $\boxed{x[0] = \lim_{z\to\infty} X(z)}$

Proof: $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots.$

Obviously, as $z \to \infty$, $z^{-n} \to 0$ since $n > 0$, this proves the theorem.

Final Value Theorem:

If $x[n] \leftrightarrow X(z)$, then $\boxed{x[\infty] = \lim_{z\to 1}\left(1 - z^{-1}\right)X(z)}$

Examples

Example 8: Find the final value of

$$X(z) = \frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$$

Solution 8: $(1-z^{-1})X(z) = (1-z^{-1})\frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$

$$= (1-z^{-1})\frac{2z^{-1}}{(1-z^{-1})(1+0.5z^{-1})} = \frac{2z^{-1}}{1+0.5z^{-1}}$$

The final value theorem yields

$$y[\infty] = \lim_{z\to 1}\frac{2z^{-1}}{1-0.8z^{-1}} = \frac{2}{0.2} = 10$$

Tutorial

Tutorial 1:

(a) Find Z transform of x[n]=3δ[n]-1.5 δ[n-2]-δ[n-3]+4 δ[n-5]

(b) Prove the differentiation property of z – transform.

(c) Prove the correlation

(d) Prove the Final Value Theorem

Inverse Z-Transform

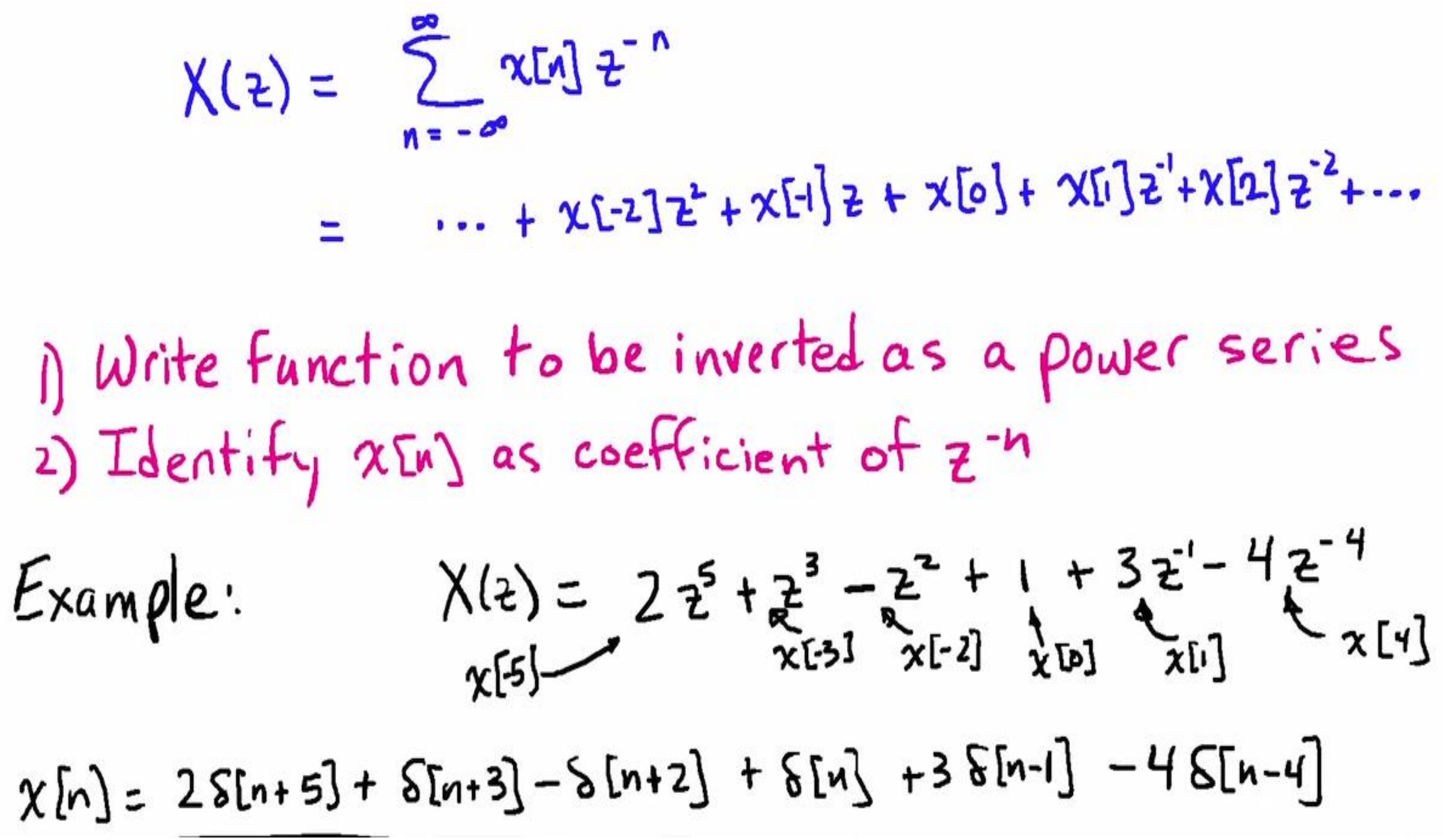
- The inverse z-Transform is defined as:

$$z^{-1}[X(z)] = x(n) = \frac{1}{2\pi j}\int_{C} X(z)z^{n-1}dz$$

- In general, the inverse z-transform may be found by using any of the following methods:
  - Power series method
  - Partial fraction method
  - Contour Integration

We won’t use this method in this course!

Power Series Expansion



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

1) Write function to be inverted as a power series

2) Identify x[n] as coefficient of z^{-n}

Example: $X(z) = \underset{x[-5]}{2z^{5}} + \underset{x[-3]}{z^{3}} - \underset{x[-2]}{z^{2}} + \underset{x[0]}{1} + \underset{x[1]}{3z^{-1}} - \underset{x[4]}{4z^{-4}}$

$x[n] = 2\delta[n+5] + \delta[n+3] - \delta[n+2] + \delta[n] + 3\delta[n-1] - 4\delta[n-4]$

Power Series Expansion

Power series expansion can invert transcendental functions of z

Ex. $X(z) = \exp\{-2z^{-1}\}$

Recall $\exp\{x\} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so

$$X(z) = \sum_{n=0}^{\infty} \frac{(-2z^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} z^{-n}$$

$$x[n] = \frac{(-2)^n}{n!} u[n]$$

Power Series Expansion

Can also invert rational $X(z)$ with long division

Ex. $X(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}$

$|z| > 1/2$

$$
\begin{array}{r|l}
 & 1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} - \frac{1}{16}z^{-4} \cdots \\
\hline
1 - \frac{1}{2}z^{-1} & 1 - z^{-1} \\
 & 1 - \frac{1}{2}z^{-1} \\
\cline{2-2}
 & -\frac{1}{2}z^{-1} \\
 & -\frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} \\
\cline{2-2}
 & -\frac{1}{4}z^{-2} \\
 & -\frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} \\
\cline{2-2}
 & -\frac{1}{8}z^{-3} \\
 & -\frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} \\
\cline{2-2}
 & -\frac{1}{16}z^{-4}
\end{array}
$$

$$
x[n] = \begin{array}{ll}
0 & n < 0 \\
1 & n = 0 \\
-1/2 & n = 1 \\
-1/4 & n = 2 \\
-1/8 & n = 3 \\
-1/16 & n = 4 \\
\vdots &
\end{array}
$$

$$x[n] = 2\,\delta[n] - (1/2)^n u[n]$$

Power Series Method

Example 1: Determine inverse z-transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

Solution 1: **By dividing the numerator of X(z) by its denominator, we obtain the power series**

$$= 1+\tfrac{3}{2}z^{-1}+\tfrac{7}{4}z^{-2}+\tfrac{15}{8}z^{-3}+\tfrac{31}{16}z^{-4}+\ldots$$

$\therefore$ x[n] = [$\underline{1}$, 3/2, 7/4, 15/8, 31/16,….]

Power Series Method

Example 2: Determine the inverse z-transform of

$$X(z) = \frac{4 - z^{-1}}{2 - 2z^{-1} + z^{-2}}$$

Solution 2: By dividing the numerator of X(z) by its denominator, we obtain the power series

$$\therefore \quad x[n] = [\, \underset{\uparrow}{\underline{2}}, \quad 1.5, \quad 0.5, \quad -0.25, \; \ldots..]$$

Power Series Method

- Tutorial1: Find the inverse z-transform of the following by power series method.

$$a)X(z)=\frac{1}{2-4z^{-2}+6z^{-3}}$$

$$b)W(z)=\frac{0.5(1-2z^{-1})(1+0.5z^{-1})}{(1+0.5z^{-1})(1-0.5z^{-1})}$$

Example1: Partial Fraction

- Decompose the following:

$$\frac{x^5-2x^4+x^3+x+5}{x^3-2x^2+x-2}$$

The numerator is of degree 5; the denominator is of degree 3. So first I have to do the long division:

$$\begin{array}{rl}
 & x^2 \\
x^3-2x^2+x-2 & \overline{)\,x^5-2x^4+x^3+0x^2+x+5} \\
 & \underline{x^5-2x^4+x^3-2x^2} \\
 & 2x^2+x+5
\end{array}$$

The long division rearranges the rational expression to give me:

$$x^2+\frac{2x^2+x+5}{x^3-2x^2+x-2}$$

Now I can decompose the fractional part. The denominator factors as $(x^2+1)(x-2)$.

$$x^2+\frac{2x^2+x+5}{\left(x^2+1\right)\left(x-2\right)}$$

Example1: Partial Fraction

The $x^2 + 1$ is irreducible, so the decomposition will be of the form:

$$\frac{2x^2+x+5}{x^3-2x^2+x-2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

Multiplying out and solving, I get:

$$2x^2 + x + 5 = A(x^2 + 1) + (Bx + C)(x - 2)$$

$x = 2$: $8 + 2 + 5 = A(5) + (2B + C)(0)$, $15 = 5A$, and $A = 3$

$x = 0$: $0 + 0 + 5 = 3(1) + (0 + C)(0 - 2)$,

$5 = 3 - 2C$, $2 = -2C$, and $C = -1$

$x = 1$: $2 + 1 + 5 = 3(1 + 1) + (1B - 1)(1 - 2)$,

$8 = 6 + (B - 1)(-1) = 6 - B + 1$,
$8 = 7 - B$, $1 = -B$, and $B = -1$

Then the complete expansion is:

$$x^2 + \frac{3}{x-2} + \frac{-x-1}{x^2+1} = x^2 + \frac{3}{x-2} - \frac{x+1}{x^2+1}$$

Example2: Partial Fraction

- Find the partial-fraction decomposition of the following expression:

$$\frac{x^2+1}{x(x-1)^3}$$

The factor $x - 1$ occurs three times in the denominator. I will account for that by forming fractions containing increasing powers of this factor in the denominator, like this:

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x}$$

Now I multiply through by the common denominator to get:

$$x^2 + 1 = Ax(x-1)^2 + Bx(x-1) + Cx + D(x-1)^3$$

I could use a system of equations to solve for A, B, C, and D, but the other method seemed easier. The two zeroing numbers are $x = 1$ and $x = 0$: so

$x = 1$: $1 + 1 = 0 + 0 + C + 0$, so $C = 2$
$x = 0$: $1 = 0 + 0 + 0 - D$, so $D = -1$

But what do I do now? I have two other variables, namely A and B, for which I need values. But since I've got values for C and D, I can pick any two other x-values, plug them in, and get a system of equations that I can solve for A and B. The particular x-values I choose aren't important, so I'll pick smallish ones:

Example2: Partial Fraction

$x = 2$:

$$(2)^2 + 1 = A(2)(2-1)^2 + B(2)(2-1) + (2)(2) + (-1)(2-1)^3$$
$$4 + 1 = 2A + 2B + 4 - 1$$
$$5 = 2A + 2B + 3$$
$$1 = A + B$$

$x = -1$:

$$(-1)^2 + 1 = A(-1)(-1-1)^2 + B(-1)(-1-1) + (2)(-1) + (-1)(-1-1)^3$$
$$1 + 1 = -4A + 2B - 2 + 8$$
$$2 = -4A + 2B + 6$$
$$2A - B = 2$$

I'm still stuck solving a system of equations, but by using the easier method to solve for C and D, I now have a simpler system to solve. Adding the two equations, I get $3A = 3$, so $A = 1$. Then $B = 0$ (so that term in the expansion "vanishes"), and the complete decomposition is:

$$\frac{1}{x-1} + \frac{2}{(x-1)^3} - \frac{1}{x}$$

Example3: Partial Fraction

- Find the partial-fraction decomposition of the following:

$$\frac{x-3}{x^3+3x}$$

Factoring the denominator, I get $x(x^2 + 3)$. I can't factor the quadratic bit, so my expanded form will look like this:

$$\frac{x-3}{x\left(x^2+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

Note that the numerator for the "$x^2 + 3$" fraction is a linear polynomial, not just a constant term.

Multiplying through by the common denominator, I get:

$$x-3 = A(x^2+3)+(Bx+C)(x)$$
$$x-3 = Ax^2+3A+Bx^2+Cx$$
$$x-3 = (A+B)x^2+(C)x+(3A)$$

Example3: Partial Fraction

The only zero in the original denominator is $x = 0$, so:

$$(0) - 3 = (A + B)(0)^2 + C(0) + 3A$$
$$-3 = 3A$$

Then $A = -1$. Since I have no other helpful x-values to work with, I think I'll take the one value I've solved for, equate the remaining coefficients, and see what that gives me:

$$x - 3 = (-1 + B)x^2 + (C)x - 3$$
$$-1 + B = 0 \text{ and } C = 1$$
$$B = 1 \text{ and } C = 1$$

(There is no one "right" way to solve for the values of the coefficients. Use whichever method "feels" right to you on a given exercise.)

Then the decomposition is:

$$\frac{-1}{x} + \frac{x+1}{x^2+3}$$

Example4: Partial Fraction

- Set up, but do not solve, the decomposition equality for the following:

$$\frac{x^4+3x-2}{\left(x^2+1\right)^3(x-4)^2}$$

Since x^2+1 is not factorable, I'll have to use numerators with linear factors. Then the decomposition set-up looks like this:

$$\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{\left(x^2+1\right)^2}+\frac{Gx+H}{\left(x^2+1\right)^3}$$

Example5: Partial Fraction

- Decompose the following:

$$\frac{x^5-2x^4+x^3+x+5}{x^3-2x^2+x-2}$$

The numerator is of degree 5; the denominator is of degree 3. So first I have to do the long division:

$$\begin{array}{r@{}l}
 & x^2 \\ \cline{2-2}
x^3-2x^2+x-2 \big) & x^5-2x^4+x^3+0x^2+x+5 \\
 & x^5-2x^4+x^3-2x^2 \\ \cline{2-2}
 & \qquad\qquad\qquad 2x^2+x+5
\end{array}$$

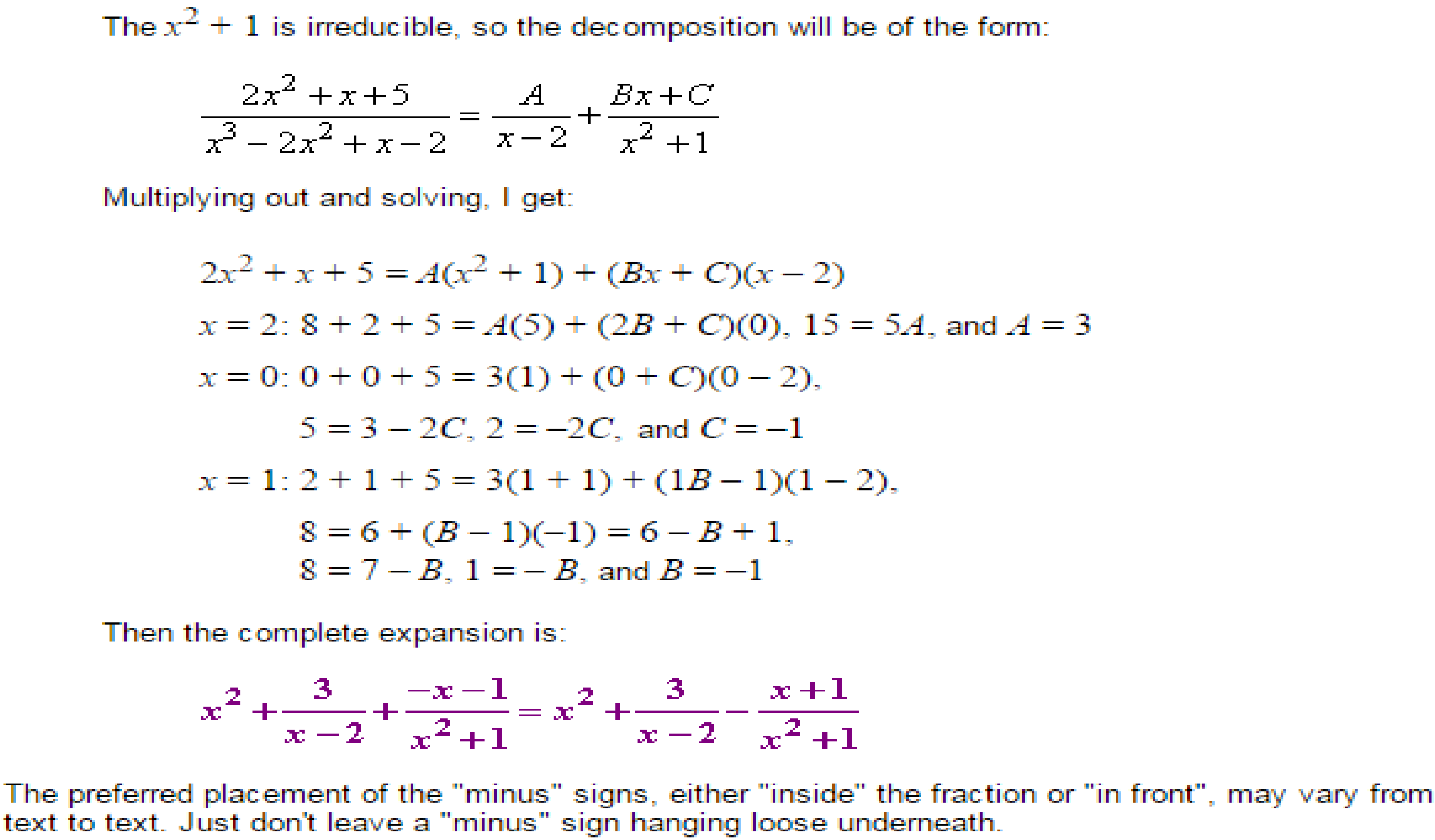
The long division rearranges the rational expression to give me:

$$x^2+\frac{2x^2+x+5}{x^3-2x^2+x-2}$$

Now I can decompose the fractional part. The denominator factors as $(x^2+1)(x-2)$.

$$x^2+\frac{2x^2+x+5}{\left(x^2+1\right)(x-2)}$$

Example5: Partial Fraction



The $x^2 + 1$ is irreducible, so the decomposition will be of the form:

$$\frac{2x^2+x+5}{x^3-2x^2+x-2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

Multiplying out and solving, I get:

$$2x^2 + x + 5 = A(x^2 + 1) + (Bx + C)(x - 2)$$

$x = 2$: $8 + 2 + 5 = A(5) + (2B + C)(0)$, $15 = 5A$, and $A = 3$

$x = 0$: $0 + 0 + 5 = 3(1) + (0 + C)(0 - 2)$,

$5 = 3 - 2C$, $2 = -2C$, and $C = -1$

$x = 1$: $2 + 1 + 5 = 3(1 + 1) + (1B - 1)(1 - 2)$,

$8 = 6 + (B - 1)(-1) = 6 - B + 1$,
$8 = 7 - B$, $1 = -B$, and $B = -1$

Then the complete expansion is:

$$x^2 + \frac{3}{x-2} + \frac{-x-1}{x^2+1} = x^2 + \frac{3}{x-2} - \frac{x+1}{x^2+1}$$

The preferred placement of the "minus" signs, either "inside" the fraction or "in front", may vary from text to text. Just don't leave a "minus" sign hanging loose underneath.

Partial Fraction Method

Example 3: Find the signal corresponding to the z-transform

$$X(z) = \frac{z^{-3}}{2 - 3z^{-1} + z^{-2}}$$

Solution 3:

$$X(z) = \frac{z^{-3}}{2-3z^{-1}+z^{-2}} = \frac{0.5}{z^3 - 1.5z^2 + 0.5z} = \frac{0.5}{z(z-1)(z-0.5)}$$

$$X(z) = 3 + \frac{1}{z} + \frac{z}{z-1} + (-4)\frac{z}{z-0.5}$$

or

$$X(z) = 3 + z^{-1} + \frac{1}{1-z^{-1}} - 4\frac{1}{1-0.5z^{-1}}$$

$$\therefore x[n] = 3\delta[n] + \delta[n-1] + u[n] - 4(0.5)^n u[n]$$

Partial Fraction Method

- Tutorial2:Find the inverse z-transform of the following by partial fraction method.

$$a)\ \frac{Y(z)}{U(z)} = \frac{z}{z^2 - 3z + 2}$$

$$b)\ X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})^2}$$

Z-Transform Solution of Linear Difference Equations

- We can use z-transform to solve the difference equation that characterizes a causal, linear, time invariant system. The following expressions are especially useful to solve the difference equations:

  - $z[y[(n-1)T] = z^{-1}Y(z) +y[-T]$
  - $Z[y(n-2)T] = z^{-2}Y(z) + z^{-1}y[-T] + y[-2T]$
  - $Z[y(n-3)T] = z^{-3}Y(z) + z^{-2}y[-T] + z^{-1}y[-2T] +y[-3T]$

Z-Transform Solution of Linear Difference Equations

Example 5: Consider the following difference equation:

$$y[nT] - 0.1y[(n-1)T] - 0.02y[(n-2)T] = 2x[nT] - x[(n-1)T]$$

where the initial conditions are y[-T] = -10 and y[-2T] = 20. Y[nT] is the output and x[nT] is the unit step input.

Solution 5: Computing the z-transform of the difference equation gives

$$Y(z) - 0.1[z^{-1}Y(z) + y[-T]] - 0.02[z^{-2}Y(z) + z^{-1}y[-T] + y[-2T]] = 2X(z) - z^{1}X(z)$$

Substituting the initial conditions we get

$$Y(z) - 0.1z^{-1}Y(z) + 1 - 0.02z^{-2}Y(z) - 0.2z^{-1} - 0.4 = (2 - z^{-1})X(z)$$

Z-Transform Solution of Linear Difference Equations

$$\left(1-0.1z^{-1}-0.02z^{-2}\right)Y(z)=\left(2-z^{-1}\right)\frac{1}{1-z^{-1}}-0.2z^{-1}-0.6$$

$$Y(z)\left[1-0.2z^{-1}-0.02z^{-2}\right]=\frac{2-z^{-1}}{1-z^{-1}}-0.2z^{-1}-0.6$$

$$Y(z)=\frac{1.4-0.6z^{-1}+0.2z^{-2}}{\left(1-z^{-1}\right)\left(1-0.1z^{-1}-0.02z^{-2}\right)}=\frac{1.4-0.6z^{-1}+0.2z^{-2}}{\left(1-z^{-1}\right)\left(1-0.2z^{-1}\right)\left(1+0.1z^{-1}\right)}$$

$$=\frac{1.4z^{3}-0.6z^{2}+0.2z}{(z-1)(z-0.2)(z+0.1)}$$

$$\frac{Y(z)}{z}=\frac{1.136}{z-1}+\frac{-0.567}{z-0.2}+\frac{0.830}{z+0.1}$$

$$Y(z)=1.136\frac{1}{1-z^{-1}}-0.567\frac{1}{1-0.2z^{-1}}+0.830\frac{1}{1+0.1z^{-1}}$$

and the output signal y[nT] is

$$y[nT]=1.136\,u[nT]-0.567\,(0.2)^{n}u[nT]+0.830\,(-0.1)^{n}u[nT]$$

Tutorial

- Tutorial3: Determine the step response of the system

$$y[n]=ay[n-1]+x[n] \quad -1<a<1$$

When the initial condition is $y[-1]=1$

End of Chapter